PHYSICAL REVIEW E VOLUME 57, NUMBER 3 MARCH 1998

Statistical mechanical approach to Fukui-Ishibashi traffic flow models

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(Received 8 July 1997)

We consider cellular automaton models for one-dimensional traffic flow problems. Starting with a microscopic relation for the updating rule describing the occupancy on each site of the road, a macroscopic evolution relation for the average speed of cars can be obtained by carrying out statistical averages. Mean field equations are obtained by considering the asymptotic form of the evolution relation. This gives the average car speed in the long time limit as a function of the car density. The evolution relation is a nonlinear mapping between the average speeds at two consecutive time steps. The mean field results can be obtained by studying the attractors of the mapping. The approach is applied to study the model recently proposed by Fukui and Ishibashi. Our calculations show that for models in which the maximum speed of each car is M, a decoupling scheme retaining correlations up to $M+1$ sites can be applied to the calculation of spatial correlations involving more than $M+1$ sites. Exact results are obtained using our approach for models without random delay. For models with random delay, results are in good agreement with simulation results. [S1063-651X(98)00403-6]

PACS number(s): 64.60.Ak, 05.40.+j, 05.70.Jk, 89.40.+k

I. INTRODUCTION

Traffic flow is a complex system. The usual method of studying these problems is to perform numerical simulations using high-speed parallel computing. Cellular automaton models become increasingly important in traffic flow problems due to their simplicity, and easiness in implementation in computers [1]. These models capture the nonlinear behavior of traffic flow problems. For example, the two-dimensional Biham-Middleton-Levine (BML) cellular automaton model gives the transition between a moving phase and a jamming phase in the traffic of a city [2]. These models can also be easily modified to deal with the effects of realistic traffic conditions. Cellular automaton models have become the major approach in studying traffic flow problems.

An important question in one-dimensional traffic flow on a highway deals with the optimum car density for achieving maximum traffic flux. The basic relation in traffic engineering is the one relating the flux or the average speed of cars to the car density on the road [3]. These relations can be obtained by measurements on actual traffic simulations. For theoretical models, the basic relation can be obtained by numerical simulations. For some simple models, one can obtain the relation analytically using qualitative arguments within some reasonable approximations. These mean field approaches provide us with some physical understanding of the traffic models [4–7]. It is useful to derive mean field theories that give the basic relation in quantitative agreement with actual situation and with numerical simulations, as they can be used for analytical predictions in traffic control. However, it is usually difficult to obtain mean field theories with quantitative accuracy. For cellular automaton traffic flow problems, successful mean field theory has been proposed for the simplest one-dimensional case in which the maximum speed $v_{\max}$ of a car at a time is 1. For models with acceleration as proposed by Nagel and Schreckenberg (NS) [8,9] and with random delay as proposed by Fukui and Ishibashi (FI) [10], numerical results are only available for $v_{\max}>1$ cases. For the two-dimensional BML model, Nagatani proposed a mean field theory, that is only in qualitative agreement with simulation data. Wang, Woo, and Hui proposed an improved mean field theory with results in better agreement with numerical results [11,12]. However, the previous mean field theories are phenomenological. They were not derived using statistical mechanical approaches starting from the microscopic details of the models. In general, the agreement between these mean field theories and numerical simulations is not so satisfactory.

Our aim is to study traffic flow problems from a microscopic point of view. Starting from the microscopic updating rule of a Boolean variable related to the occupancy defined on each site, the average speed at time t can be obtained in terms of these variables. Systematic statistical mechanical treatment can be carried out. A decoupling scheme is introduced to deal with the spatial correlation functions involving averages over products of Boolean variables on a string of neighboring sites. A macroscopic dynamical mapping is thus obtained for the average speed. Mean field results can be obtained by studying the attractors of the nonlinear dynamical system defined by the mapping.

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The microscopic approach in deriving macroscopic results is in line with the viewpoint of statistical mechanics and nonlinear dynamics. Such an approach is useful in our understanding of traffic flow systems. In Refs. [13,14], we studied the NS model with $v_{\max}=1$ as a first attempt in deriving results within such an approach. In this paper, we shall further study the FI model [10] of high-speed cars, i.e., $v_{\max}>1$. For the $v_{\max}=2$ FI model, a dynamical relation between the average speeds $V(t)$ and $V(t+1)$ is obtained. For the $v_{\max}=2$ model without delay, we obtained results in exact agreement with numerical results. For the model with random delay, mean field results are obtained by studying the attractors of the dynamical mapping. Results with an arbitrary degree of random delay are in good qualitative as well as quantitative agreement with simulation data. The approach in this work can be generalized to treat the one-dimensional NS model, the two-dimensional BML model, and other models with realistic traffic situations incorporated.

II. MODELS AND EVOLUTION EQUATIONS

The basic cellular automaton model describing a one-dimensional traffic flow problem is rule 184, as classified by Wolfram [1]. Within the model, cars are treated as distinguishable particles. The road is modeled by a discrete lattice. Each lattice site can be occupied by at most one car. The system evolves in discrete time steps. At each time step, each particle on the road moves forward one site to the right, say, if the site on the right is not occupied by another car at the previous time step. If the site in front is occupied, the car will remain stationary even if the blocking car moves away from the site in front. This model, though simple, exihibits the phenomenon of transition between a moving phase with maximum possible speed to a partially jamming phase with slower speed. As a generalization of the basic rule 184, Fukui and Ishibashi [10] considered a model in which a car may have maximum speed $v_{\max}=M$ $(M>1)$. At each time step, if there are M or more empty sites in front of a car, the car moves forward M lattice sites. If there are only N $(N<M)$ empty sites in front of the car, the car moves forward N sites. Furthermore, a model with random delay can be introduced in the following way. When a car can move forward by M sites, there is a probability f that the car will slow down and move forward by only $M-1$ sites. It should be pointed out that the acceleration mechanism in the FI model is different from that of the NS model, and that each of these models reflects part of the actual situation in realistic traffic problems.

Let $s_i(t)$ be a Boolean variable describing the occupancy of the ith lattice site at time t. The variable takes on $s_i(t)=1$ if the site is occupied and $s_i(t)=0$ if the site is empty. Rule 184 can be expressed as

$$s_i(t+1)=s_i(t)s_{i+1}(t)+s_{i-1}(t)\overline{s}_i(t), \tag{1}$$

where $\overline{s}_i(t)\equiv 1-s_i(t)$ is the conjugate of the Boolean variable $s_i(t)$. The variables $s_i(t)$ and $\overline{s}_i(t)$ have the properties

$$s_i(t)s_i(t)=s_i(t), \quad s_i(t)\overline{s}_i(t)=0. \tag{2}$$

For the FI model with $v_{\max}=M$ and random delay, the updating rule is given by

$$\begin{aligned}
s_i(t+1)={}&s_i(t)s_{i+1}(t)+s_{i-1}(t)\overline{s}_i(t)s_{i+1}(t)\\
&+s_{i-2}(t)\overline{s}_{i-1}(t)\overline{s}_i(t)s_{i+1}(t)+\cdots\\
&+s_{i-M+1}(t)\overline{s}_{i-M+2}(t)\cdots\overline{s}_{i-1}(t)\overline{s}_i(t)s_{i+1}(t)\\
&+\theta_{i-M+1,t}(f)s_{i-M+1}(t)\overline{s}_{i-M+2}(t)\\
&\times\overline{s}_{i-M+3}(t)\cdots\overline{s}_{i-1}(t)\overline{s}_i(t)\overline{s}_{i+1}(t)\\
&+\theta_{i-M,t}(1-f)s_{i-M}(t)\overline{s}_{i-M+1}(t)\overline{s}_{i-M+2}(t)\\
&\times\cdots\overline{s}_{i-2}(t)\overline{s}_{i-1}(t)\overline{s}_i(t),
\end{aligned} \tag{3}$$

where $\theta_{i,t}(f)$ is a random delay factor of the ith site at time t. It takes on the value unity with probability f, and vanishes with probability $\overline{f}\equiv 1-f$. The factor $\theta_{i,t}(f)$ can be called a stochastic Boolean variable, and its corresponding conjugate is $\overline{\theta_{i,t}(f)}\equiv 1-\theta_{i,t}(f)=\theta_{i,t}(\overline{f})$, which takes on the value unity with probability $1-f$ and vanishes with probability f. The variables on the same site at the same time satisfy the relations

$$\theta_{i,t}(f)\theta_{i,t}(f)=\theta_{i,t}(f), \quad \theta_{i,t}(f)\overline{\theta_{i,t}(f)}=0. \tag{4}$$

To illustrate our approach, we consider the FI model with $M=2$. It is straightforward to generalize our method to cases with higher values of M. The basic updating rule becomes

$$\begin{aligned}
s_i(t+1)={}&s_i(t)s_{i+1}(t)+s_{i-1}(t)\overline{s}_i(t)\overline{s}_{i+1}(t)\\
&+\theta_{i-1,t}(f)s_{i-1}(t)\overline{s}_i(t)\overline{s}_{i+1}(t)\\
&+\theta_{i-2,t}(1-f)s_{i-2}(t)\overline{s}_{i-1}(t)\overline{s}_i(t).
\end{aligned} \tag{5}$$

Let L be the total number of sites on the road and N be the total number of cars. The average speed at time t can be expressed microscopically as

$$\begin{aligned}
V(t)={}&\frac{1}{p}[\langle s_i(t)\overline{s}_{i+1}(t)\rangle\\
&+\langle\theta_{i,t}(1-f)s_i(t)\overline{s}_{i+1}(t)\overline{s}_{i+2}(t)\rangle],
\end{aligned} \tag{6}$$

where $p=N/L$ is the car density. The angular bracket $\langle\cdots\rangle=(1/L)\Sigma_{i=1}^{L}(\cdots)$ represents the spatial average of the microscopic quantities in the curly bracket. This type of averages gives the spatial correlation over a number of neighboring sites.

We note that nonvanishing contributions to the average $\langle\cdots\rangle$ come only from terms where $s_j(t)$ takes on the value unity, and where $\overline{s}_k(t)$ takes on unity, and hence $s_k(t)$ takes on zero. Hence we introduce the following simplified notations to denote the averages:

$$\begin{aligned}
\langle s_i(t)\overline{s}_{i+1}(t)\rangle&=\langle 10\rangle_t,\\
\langle s_i(t)s_{i+1}(t)\rangle&=\langle 11\rangle_t,\\
\langle s_i(t)\overline{s}_{i+1}(t)\overline{s}_{i+2}(t)\rangle&=\langle 100\rangle_t.
\end{aligned} \tag{7}$$

etc. For the random variables $\theta_{i,t}(f)$ there are nonvanishing contributions only when they take on the value unity. Therefore, they can be taken outside the angular bracket and replaced simply by a factor f. Equation (6) for the microscopic expression of the average speed at time t can then be rewritten as

$$V(t)=\frac{1}{p}[\langle 10\rangle_t+(1-f)\langle 100\rangle_t]. \tag{8}$$

We can also write down the average speed at time $t+1$ from Eq. (6) as

$$V(t+1)=\frac{1}{p}\langle s_i(t+1)\overline{s}_{i+1}(t+1)[2-\theta_{i,t+1}(f)-\theta_{i,t+1}(1-f)s_{i+2}(t+1)]\rangle, \tag{9}$$

Substituting Eq. (5) into Eq. (9), and using the relations of the Boolean variables given by Eqs. (2)–(4), we obtain the average speed at time $t+1$ expressed in the simplified notations as

$$\begin{aligned}V(t+1)=2-f-\frac{1}{p}\{&(2-f)[\langle 111\rangle_t+\langle 1011\rangle_t\\&+(1-f)\langle 10\,011\rangle_t]+(1-f)[\langle 1101\rangle_t\\&+\langle 10\,101\rangle_t+f(\langle 1100\rangle_t+\langle 10\,011\rangle_t\\&+\langle 10\,100\rangle_t)+(1-f)(\langle 100\,101\rangle_t+\langle 100\,011\rangle_t)\\&+f(1-f)\langle 100\,100\rangle_t]\}.\end{aligned} \tag{10}$$

Equations (8) and (10) give the relation between $V(t)$ and $V(t+1)$ at two consecutive time steps. For $f=1$, the FI model reduces to the $v_{\max}=1$ case, which corresponds to the basic rule-184 model. The equations become

$$V(t)=\frac{1}{p}\langle 10\rangle_t,\quad V(t+1)=1-\frac{1}{p}(\langle 111\rangle_t+\langle 1011\rangle_t). \tag{11}$$

For $f=0$, Eqs. (8) and (10) give the results corresponding to the $v_{\max}=2$ FI model without random delay. The equations become

$$V(t)=\frac{1}{p}(\langle 10\rangle_t+\langle 100\rangle_t),$$

$$\begin{aligned}V(t+1)=2-\frac{1}{p}[&2(\langle 111\rangle_t+\langle 1011\rangle_t+\langle 10\,011\rangle_t)+\langle 1101\rangle_t\\&+\langle 10\,101\rangle_t+\langle 100\,101\rangle_t+\langle 100\,011\rangle_t].\end{aligned} \tag{12}$$

III. DECOUPLING OF SPATIAL CORRELATION FUNCTIONS

In order to derive a relation between $V(t)$ and $V(t+1)$, we need to treat the spatial correlation functions which are averages of the Boolean variables $s_i(t)$ over a string of neighboring sites. For the FI model without random delay and when $v_{\max}=1$, the two-site correlation function $\langle 10\rangle_t$ can be obtained from Eq. (11) as $\langle 10\rangle_t=pV(t)$. From this we have

$$\langle 11\rangle_t=\langle 1\rangle_t-\langle 10\rangle_t=p[1-V(t)],$$
$$\langle 00\rangle_t=1-p-pV(t). \tag{13}$$

The average $\langle 110\rangle_t$, which involves more than two sites, can be treated according to probability theory as [9] $\langle 110\rangle_t=\langle 11\rangle_t\,P(11|0)_t$, where $P(11|0)_t$ is the conditional probability that the $(i+2)$th site is empty given that the ith and $(i+1)$th sites are occupied. Since each particle can only move forward one site at a time for the $v_{\max}=1$ model, the influence on the variable $s(t)$ at a site at time t comes only from the two neighboring sites. Therefore, we may ignore correlations involving more than two sites and retain only two-site correlations. We have

$$P(11|0)_t=P(1|0)_t=\frac{\langle 10\rangle_t}{\langle 1\rangle_t}=V(t), \tag{14}$$

It follows that

$$\langle 110\rangle_t=pV(t)[1-V(t)]. \tag{15}$$

For the $v_{\max}=2$ model without random delay, each particle can at most move forward two sites at a time. In this case, we cannot simply ignore correlations involving more than two sites. Not only does the change in the variable $s_i(t)$ at a site at time t depend on the states of the neighboring sites, it also depends on the states of the next nearest neighbors. Therefore we should treat three-site correlations more accurately, and decouple correlations involving four or more sites. Equation (12) does not uniquely determine $\langle 10\rangle_t$ and $\langle 100\rangle_t$. To treat the three-site correlation functions, we follow $\langle 10\rangle_t=pV(t)$ and propose the approximations that $\langle 10\rangle_t=pu$ and $\langle 100\rangle_t=pu^2$, where u is a parameter related to the average speed $V(t)$ and remains to be determined. It follows that $\langle 11\rangle_t=p(1-u)$ and $\langle 00\rangle_t=1-p-pu$. The other three-site correlation functions can be expressed in terms of u as

$$\begin{aligned}\langle 101\rangle_t=\langle 110\rangle_t&=pu(1-u),\\ \langle 010\rangle_t=\langle 001\rangle_t=\langle 100\rangle_t&=pu^2,\\ \langle 111\rangle_t&=p(1-u)^2.\end{aligned} \tag{16}$$

For the $v_{\max}=2$ model with random delay, we need to consider the effects of the factor f, which describes the degree of randomness. To treat multisite correlation functions, we assume that the case for given f can be approximated as a simple average of the two limiting cases corresponding to $f=0$ and 1. According to this assumption,

$$\langle 100\rangle_t=pu\left[(1-f)u+f\left(1-\frac{pu}{1-p}\right)\right]. \tag{17}$$

The other correlation functions are treated within the approximation discussed in the models without random delay. Thus we still have $\langle 10\rangle_t=pu$ and $\langle 110\rangle_t=pu(1-u)$. It follows that

$$\langle 11\rangle_t = p(1-u),$$

$$\langle 00\rangle_t = 1-p-pu,$$

$$\langle 111\rangle_t = \langle 11\rangle_t - \langle 110\rangle_t = p(1-u)^2,$$

$$\langle 010\rangle_t = \langle 10\rangle_t - \langle 110\rangle_t = pu^2, \tag{18}$$

$$\langle 101\rangle_t = \langle 10\rangle_t - \langle 100\rangle_t = pu\left[1-(1-f)u-f\frac{1-p-pu}{1-p}\right],$$

$$\langle 000\rangle_t = \langle 00\rangle_t - \langle 100\rangle_t = 1-p-pu-pu\left[(1-f)u+f\frac{1-p-pu}{1-p}\right],$$

$$\langle 001\rangle_t = \langle 00\rangle_t - \langle 000\rangle_t = \langle 100\rangle_t = pu\left[(1-f)u+f\left(1-\frac{pu}{1-p}\right)\right].$$

It should be noted that the above results reduce to those of the $v_{\max}=2$ model without random delay when $f=0$, and reduce to those of the $v_{\max}=1$ model without delay when $f=1$. For $f=1$, the relations $\langle 100\rangle_t = pu[1-pu/(1-p)]$ and $\langle 000\rangle_t = (1-p-pu)^2/(1-p)$ are consistent with the decoupling scheme of retaining only two-site correlation functions, i.e.,

$$\langle 100\rangle_t = \frac{\langle 10\rangle_t\langle 00\rangle_t}{\langle 0\rangle_t} = pu\left(1-\frac{pu}{1-p}\right),$$

$$\langle 000\rangle_t = \frac{\langle 00\rangle_t\langle 00\rangle_t}{\langle 0\rangle_t} = \frac{(1-p-pu)^2}{1-p}. \tag{19}$$

Starting with Eqs. (17) and (18), we can decouple the correlation functions involving more than three sites as follows:

$$\langle 1011\rangle_t = \langle 101\rangle_t \frac{\langle 011\rangle_t}{\langle 01\rangle_t} = pu(1-u)\left[1-(1-f)u-f\frac{1-p-pu}{1-p}\right],$$

$$\langle 1101\rangle_t = \frac{\langle 110\rangle_t\langle 101\rangle_t}{\langle 10\rangle_t} = \langle 1011\rangle_t,$$

$$\langle 1100\rangle_t = \langle 110\rangle_t - \langle 1101\rangle_t = pu(1-u)\left[(1-f)u+f\frac{1-p-pu}{1-p}\right],$$

$$\langle 10\,011\rangle_t = \langle 100\rangle_t P(00|1)_t P(01|1)_t = \langle 100\rangle_t \frac{\langle 001\rangle_t}{\langle 00\rangle_t}\frac{\langle 011\rangle_t}{\langle 01\rangle_t} = p^2u^2(1-u)\frac{(1-f)u+f(1-p-pu)/(1-p)}{1-p-pu},$$

$$\langle 100\,011\rangle_t = \langle 100\rangle_t P(00|0)_t P(00|0)_t P(01|1)_t = p^2u^2(1-u)\left[(1-f)u+f\frac{1-p-pu}{1-p}\right]^2 \times \frac{1-p-pu-pu[(1-f)u+f(1-p-pu)/(1-p)]}{(1-p-pu)^2}, \tag{20}$$

$$\langle 100\,101\rangle_t = \langle 100\rangle_t P(00|1)_t P(01|0)tP(10|1)_t = p^2u^3(1-u)[(1-f)u + f(1-p-pu)/(1-p)]^2 \times \frac{1-(1-f)u-f(1-p-pu)/(1-p)}{1-p-pu},$$

$$\langle 10\,101\rangle_t = \langle 101\rangle_t P(01|0)_t P(10|1)_t = pu^2(1-u)\frac{[fpu+(1-f)(1-p)(1-u)]^2}{(1-p)^2},$$

$$\langle 10\,100\rangle_t = \langle 1010\rangle_t - \langle 10101\rangle_t = pu^2[1-(1-f)u-f(1-p-pu)/(1-p)] \times [(1-f)u+f(1-p-pu)/(1-p)],$$

$$\langle 100\,100\rangle_t = \frac{\langle 100\rangle_t\langle 001\rangle_t\langle 010\rangle_t\langle 100\rangle_t}{\langle 00\rangle_t\langle 01\rangle_t\langle 10\rangle_t} = p^2u^3\frac{[(1-f)u+f(1-p-pu)/(1-p)]^3}{1-p-pu}.$$

IV. NONLINEAR MAPPING BETWEEN AVERAGE SPEEDS AND MEAN FIELD EQUATIONS

Substituting the results [Eq. (20)] for the multisite correlations into the expressions of $V(t)$ and $V(t+1)$ in Eqs. (8) and (10), we obtain

$$V(t) = u[1+(1-f)U_f],$$

$$\begin{aligned} V(t+1) = {} & (2-f)u\{1+U_f(1-u)[1-(1-f)U_fU_p]\} \\ & -(1-f)u\{(1-U_f)(1-uU_f)+fU_f(1-uU_fU_p) \\ & +(1-f)U_f^2U_p(1-U_f[(1-f)u+U_p(1-u)])\}, \end{aligned} \tag{21}$$

where

$$U_f = (1-f)u+f\frac{1-p-pu}{1-p},$$

$$U_p = \frac{pu}{1-p-pu}. \tag{22}$$

The parameter u can be written explicitly in terms of f, p, and $V(t)$ as

$$u=\frac{\{(1+f-f^2)^2+4(1-f)[1-f/(1-p)]v(t)\}^{1/2}-(1+f-f^2)}{2(1-f)[1-f/(1-p)]}. \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (21), we obtain a nonlinear mapping $V(t)\rightarrow V(t+1)$ between the macroscopic average speeds at two consecutive time steps. The attractor of the mapping gives the mean field result of the average speed in the asymptotic steady state. Figure 1 shows the results of our approach for the average speed as a function of car density p for different values of f for the $v_{\max}=2$ model. Results are also compared with those obtained by numerical simulations [10]. Results are in good quantitative agreement except in the vicinity of the critical point corresponding to the transition from a moving phase to a partially jamming phase. This slight discrepancy can be attributed to the finite size of the system used in the simulations, and the finite time used in the simulations as well as a critical slowing down. It should, however, be pointed out that our results for $f\rightarrow 0$ and $f\rightarrow 1$ give the exact mean field results for the case of $v_{\max}=2$ and $v_{\max}=1$ models without delay, respectively.

For $f\rightarrow 1$, the nonlinear mapping of the average speed [Eq. (21)] is

$$V(t+1)=2V(t)-\frac{1}{1-p}V^2(t)+\frac{p}{1-p}V^3(t). \tag{24}$$

The fixed point $v=V(t\rightarrow\infty)$ in the long time limit satisfies the mean field equation

$$v=2v-\frac{1}{1-p}v^2+\frac{p}{1-p}v^3. \tag{25}$$

The two nonvanishing fixed points are $v_1=1$ and $v_2=1/p-1$. The solution v_1 is stable for p in the range $0<p<\frac{1}{2}$,

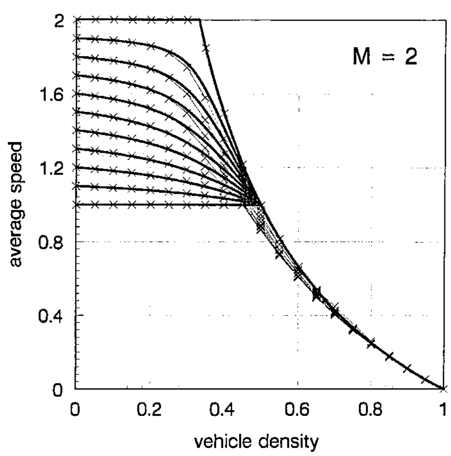

FIG. 1. The fundamental diagram of the Fukui-Ishibashi traffic flow model for the $M=2$ high-speed car with stochastic delay. The bold solid curves represent the simulation results. The crosses ($\times$) and the thin line connecting them represent the theoretical calculation results from the mean field equations given in this paper by the statistical mechanical approach. The 11 curves from top to bottom correspond to delay probabilities $f=0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9$, and 1.0, respectively.

and v_2 is stable in the range $\frac{1}{2}<p<1$. These give the exact result for the $v_{\max}=1$ model without delay as

$$V(t\rightarrow\infty)=\begin{cases}1, & 0\leqslant p\leqslant\dfrac{1}{2}\\[2ex] \dfrac{1}{p}-1, & \dfrac{1}{2}\leqslant p\leqslant 1.\end{cases} \tag{26}$$

Although this result has been reported in the literature [2,6,8,9], our approach is quite different. We started with a microscopic equation for the local updating rule. By statistical averaging, we derived a nonlinear mapping describing the macroscopic time evolution. The attractors and their corresponding basins give the exact average car speed in the long time limit as a function of the car density.

For $f\rightarrow 0$, the nonlinear mapping of the average speed [Eq. (21)] becomes

$$V(t)=u^2+u,$$

$$V(t+1)=2-(1-u)\left[(1+u)(1-u)(2+u)+\frac{pu^4[3-p+(1-p)u-pu^2]}{1-p-pu}\right]. \tag{27}$$

The fixed points of this dynamical mapping satisfy the equation

$$u^2+u=2-(1-u)\left\{(1+u)(1-u)(2+u)+\frac{pu^4[3-p+(1-p)u-pu^2]}{1-p-pu}\right\}. \tag{28}$$

The two nonvanishing solutions are $u_1=1$ and $u_2+(u_2)^2=1/p-1$. They correspond to the asymptotic average speeds $v_1=2$ and $v_2=1/p-1$, respectively. The corresponding ranges of car density to which v_1 and v_2 apply can be determined by the basins of the attractors. The conditions of stability is $|\partial v(t+1)/\partial v(t)|\leqslant 1$. The range of stability for $v_1=2$ is $p<\frac{1}{3}$; while the range of stability for $v_2=1/p-1$ is $p>\frac{1}{3}$. These give the exact result for the $v_{\max}=2$ model without delay as

$$V(t\rightarrow\infty)=\begin{cases}2, & 0\leqslant p\leqslant\dfrac{1}{3}\\[2ex] \dfrac{1}{p}-1, & \dfrac{1}{3}\leqslant p\leqslant 1.\end{cases} \tag{29}$$

It should be pointed out that Fukui and Ishibashi gave the mean field result of the general case of $v_{\max}=M$, using phenomenonlogical arguments, as [10]

$$V(t\to\infty)=\begin{cases} M, & 0\leqslant p\leqslant \dfrac{1}{M+1} \\ \dfrac{1}{p}-1, & \dfrac{1}{M+1}\leqslant p\leqslant 1. \end{cases} \tag{30}$$

This result agrees with numerical simulations. However, the derivation presented in Ref. [10] is not rigorous, and hence our approach complement that of Ref. [10]. Our approach can also be extended to cases with random delay.

V. SUMMARY

We studied the one-dimensional traffic flow model proposed by Fukui and Ishibashi. The state at each site is characterized by a Boolean variable. The updating rule can be described by a microscopic relation between $s_i(t)$ and $s_i(t+1)$. The average car speed at time t can be expressed microscopically in terms of these variables. Carrying out statitistical averaging and employing a decoupling scheme, we arrived at a macroscopic evolution relation. This relation is expressed as a nonlinear mapping $V(t)\to V(t+1)$ between the average speeds at two consecutive time steps. The attractor of this mapping gives the average speed in the long time limit. The derivation involves calculations of averages over many neighboring sites. We used probability theory and a decoupling scheme to treat these multisite correlation functions. The decoupling scheme is based on the consideration that, for the $v_{\max}=M$ FI model, only the neighboring $M+1$ sites of a given site will affect the variable on that site. Hence we retain only averages involving up to $M+1$ sites and decouple those involving more than $M+1$ sites into products of averages involving $M+1$ or less sites. Our results indicate that the decoupling scheme gives exact results for models without random delay. The results agree well with numerical simulations. For models with random delay, an additional approximation is used. It amounts to assuming that the case of arbitrary f can be approximated as a simple average of the two limiting cases corresponding to $f=0$ and 1. The results obtained are in good agreement with numerical simulations performed for different values of f. Our method presented here can be generalized to study other traffic flow models such as the one-dimensional NS model and the two-dimensional BML model. Considerations reflecting other realistic traffic situations can also be incorporated.

ACKNOWLEDGMENTS

This work was supported in part by the Research Grant Council (RGC) of the Hong Kong Government through Grant No. CUHK4191/97P. One of us (B.H.W.) would like to thank the Department of Physics at the Chinese University of Hong Kong for their support. B.H.W. acknowledges the support from the National Basic Research Climbing Project ''Nonlinear Science'' and the National Natural Science Foundation of China (Grant No. 49474216).

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